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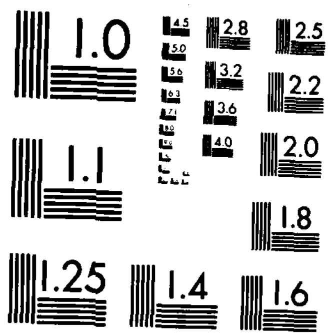
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MRC Technical Summary Report #2811

SOBOLEV-POINCARÉ'S INEQUALITY AND THE NEUMANN PROBLEM FOR $u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$

Isamu Fukuda

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

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ABSTRACT

It is shown that solutions of the Neumann problem

$$u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) \quad \text{in} \quad \Omega \times (0,T)$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0,T)$$

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega$$

tend to some constant solutions in finite time, where $1 < p < 2$ and Ω is a bounded domain in $\mathbb{R}^N$.

In order to prove this, we establish Sobolev-Poincaré's inequality for functions in $W^{1,p}(\Omega)$ under some assumptions.

We treat the extinction phenomena for the equation $u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda u$ $(\lambda > 0)$ with Neumann boundary conditions.

AMS (MOS) Subject Classifications: 35K55, 35K60

Key Words: quasilinear singular parabolic equation, Neumann problem, homogenization, Sobolev-Poincaré's inequality, extinction, boundary value problems,

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*Department of Mathematics, Kokushikan University, Setagaya Tokyo 154, Japan

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SIGNIFICANCE AND EXPLANATION

~~The~~ An equation $\cancel{u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u)}$ is used as a model for a broad class of singular and degenerate parabolic equations. The degenerate type $(p > 2)$ has been treated by many authors, but the behavior of solutions of the singular type $(1 < p < 2)$ is less well understood.

In this paper we establish the homogenization effect of the singular problem with Neumann boundary conditions, i.e. there exists a finite number $T^* > 0$ such that a solution $u(x,t)$ tends to some constant at $t \to T^*$ and $u(x,t) = \text{const.}$ for $t > T^*$.

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SOBOLEV-POINCARÉ'S INEQUALITY AND THE
NEUMANN PROBLEM FOR $u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$

Isamu Fukuda*

1. INTRODUCTION

In this paper, we will discuss the Neumann boundary value problem for the equation

$$\begin{aligned} &u_t = \mathrm{div}\,(|\nabla u|^{p-2}\nabla u)\ (\equiv \Delta_p u) && \text{in } \Omega \times (0,T) \\ &\frac{\partial u}{\partial \nu} = 0 && \text{on } \partial\Omega \times (0,T) \qquad \text{(NP)} \\ &u(x,0) = u_0(x) && \text{in } \Omega \end{aligned}$$

where $1 < p < 2$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $\frac{\partial u}{\partial \nu}$ is an outer normal derivative of u.

Consider the problem (NP) in $\Omega = (0,1)$. It takes the form

$$\begin{aligned} &u_t = (|u_x|^{p-2}u_x)_x && \text{in } (0,1) \times (0,T) \\ &u_x(0,t) = u_x(1,t) = 0 && t \in (0,T) \qquad (\mathrm{NP})_1 \\ &u(x,0) = u_0(x) && x \in (0,1) \end{aligned}$$

Differentiating formally this equation with respect to x, we obtain that $v(\equiv u_x)$ satisfies

$$\begin{aligned} &v_t = (|v|^{p-2}v)_{xx} && \text{in } (0,1) \times (0,T) \\ &v(0,t) = v(1,t) = 0 && t \in (0,T) \qquad (\mathrm{DP})_1 \\ &v(x,0) = v_0(x)\ (\equiv u_{0x}(x)) && x \in (0,T)\ . \end{aligned}$$

From Sabinina [15] and Berryman-Holland [6], we know that $v(x,t)$ tends to zero in finite time, that is, $u_x(x,t)$ tends to zero in finite time. Hence there exists a positive number T^* such that $u(x,t) = \bar{u}_0 (= \int_0^1 u_0(x)dx)$ for $t \geqq T^*$.

In other words, we obtain the interesting phenomenon that the solution becomes homogeneous in finite time.

*Department of Mathematics, Kokushikan University, Setagaya Tokyo 154, Japan

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The question comes out how much of this is shared by the Neumann problem for (NP) in general domain in $\mathbb{R}^N$. (Alikakos [1])

We will here give an answer for this problem.

In the proof, Sobolev-Poincaré's inequality for functions belonging to $W^{1,p}(\Omega)$ (not $W_0^{1,p}(\Omega)$) plays an essential role. Hence we will establish this inequality under some assumptions before proving the homogenization effect of the Neumann problem (NP).

The singular and degenerate equation (NP) with any $p > 1$ has been actively studied and is a model for a broad class of sing ar and degenerate parabolic equaitons. Existence, uniqueness and regularity results for both case $1 < p < 2$ and $p > 2$ can be found in Lions [12], di Benedetto [7] and Otani [14].

Especially the case $p > 2$ has been treated by Alikakos-Rostamian [2], [3], [4] and Alikakos-Evans [5]. In their papers, decay estimates for the gradient of the solutions in $L^p(\Omega)$ and $L^\infty(\Omega)$ have been obtained. (L^∞-estimate under the assumption that Ω is convex.) Moreover, the regularizing effect has been proved using the monotonicity of ∇u in $L^\infty(\Omega)$.

Throughout this paper, we denote L^p-norm $\|\cdot\|_{L^p(\Omega)}$ by $\|\cdot\|_p$ and abbreviate Ω in the integral $\int_\Omega \cdot\, dx$.

This work was done when the author was visiting the Mathematics Research Center at the University of Wisconsin-Madison. I wish to thank Prof. M. G. Crandall for his useful suggestion and Prof. M. Tsutsumi for his helpful comment.

2. SOBOLEV-POINCARÉ'S INEQUALITY

It is well-known that Sobolev-Poincaré's inequality holds for functions belonging to $W_0^{1,p}(\Omega)$, but we can not apply this inequality to the Neumann problem. Then we need to establish the inequality for functions belonging to $W^{1,p}(\Omega)$ under some assumptions.

Proposition 2.1

Let $\phi(s)$ be a continuous function from $\mathbb{R}$ to $\mathbb{R}$, and ϕ has only one zero at $s = 0$, that is, $\phi(s) = 0$ implies $s = 0$.

If $\int \phi(v(x))dx = 0$ and v belongs to $W^{1,p}(\Omega)$ $(p > 1)$ then

$$\|v\|_q \leq c\|\nabla v\|_p$$

for $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{N}$ and c is a constant independent of v and depends on p, q and Ω.

Proof. Assume that the statement is false. Then there exists a sequence $\{v_n\}$ such that

$$\|v_n\|_q > n\|\nabla v_n\|_p \,. \tag{2.1}$$

Without loss of generality, we can assume $\|v_n\|_q = 1$.

Since $\{v_n\}$ is bounded in $W^{1,p}(\Omega)$ and $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{N}$, we can choose a subsequence $\{v_{n'}\}$ such that $v_{n'}$ tends to v strongly in $L^q(\Omega)$ and $v \not\equiv 0$.

From (2.1), we have

$$\|\nabla v_n\|_p \leq \frac{1}{n} \,.$$

Then ∇v_n goes to zero as $n \to \infty$ and $\nabla v = 0$. Hence v is constant and not zero.

On the other hand, since $\int \phi(v(x))dx = |\Omega|\phi(v) = 0$, we can reduce that $v(x) \equiv 0$. This is a contradiction. Q.E.D.

This proposition is a generalization of Sobolev-Poincaré's inequality under the condition $\int v(x)dx = 0$. (P152, Gilbarg-Trudinger [10])

Corollary.

Let $1 < p < 2$ and $k \geq 1$.

If $\int w(x)dx = 0$ and $\nabla\left(|w|^{\frac{k-1}{p}} w\right)$ belongs to $L^p(\Omega)$, then

$$K\|w\|_{k+1}^{p+k-1} \leq \|\nabla\left(|w|^{\frac{k-1}{p}} w\right)\|_p^p \tag{2.2}$$

for $k \geq \max\left(1, \frac{1}{p}(2N - p - Np)\right)$ and K is a constant independent of w and depends on p, k and Ω.

Proof. In the proposition, taking $\phi(s) = |s|^{-\frac{k-1}{p+k-1}} s$, $v = |w|^{\frac{k-1}{p}} w$, $p = p$ and $q = \frac{p(k+1)}{p+k-1}$, we can easily obtain the corollary.

3. NEUMANN PROBLEM FOR $u_t = \Delta_p u$

Integrating the equation (NP) over $\Omega \times [0,t)$, we have

$$\int u(x,t)dx = \int u_0(x)dx \ . \tag{3.1}$$

Let $\bar{u}_0 = \frac{1}{|\Omega|}\int u_0(x)dx$ and $w(x,t) = u(x,t) - \bar{u}_0$ which satisfies

$$\int w(x,t)dx = 0$$

and

$$\left.\begin{aligned} & w_t = \Delta_p w && \text{in } \Omega \times (0,T) \ , \\ & \frac{\partial w}{\partial \nu} = 0 && \text{on } \partial\Omega \times (0,T) \ , \\ & w(x,0) = u_0(x) - \bar{u}_0 \equiv w_0 && \text{in } \Omega \ . \end{aligned}\right\} \qquad (NP)_2$$

Initially we discuss the existence and uniqueness results of the problem $(NP)_2$.

Theorem 3.1

Let w_0 be in $L^{k+1}(\Omega)$. $(k \geq 1)$ Then there exists an unique solution of $(NP)_2$ satisfying

(1) $w \in C([0,T] : L^{k+1}(\Omega)) \cap L^p(0,T : W^{1,p}(\Omega))$,

(2) $t^{1/2}w_t \in L^2(\Omega \times (0,T))$.

Since the equation is singular, we regularize the problem, obtain various estimates rigorously and then get the results by passing to the limit.

We associate with problem $(NP)_2$ the following nonsingular problem

$$\left.\begin{aligned} & w_t^\varepsilon = \operatorname{div}\left((|\nabla w^\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla w^\varepsilon\right) && \text{in } \Omega \times (0,T) \\ & \frac{\partial w^\varepsilon}{\partial \nu} = 0 && \text{on } \partial\Omega \times (0,T) \\ & w^\varepsilon(x,0) = w_0^\varepsilon(x) && \text{in } \Omega \end{aligned}\right\} \qquad (NP)_\varepsilon$$

with $\varepsilon > 0$ and $w_0^\varepsilon \in C^\infty(\bar{\Omega})$ such that $w_0^\varepsilon \to w_0$ strongly in $L^{k+1}(\Omega)$ as $\varepsilon \to 0$. By the general theory of quasilinear parabolic equation (Chapter 5, [11]), the solutions of $(NP)_\varepsilon$ are $C^\infty(\bar{\Omega} \times [0,T])$.

We have the following lemma.

Lemma 3.1

Let w^ε be a solution of $(NP)_\varepsilon$. We have

$$\|w^\varepsilon(t)\|_{k+1} \leqq C \qquad (k \geqq 1) \tag{3.2}$$

$$\|\nabla w^\varepsilon(t)\|_p \leqq C \tag{3.3}$$

$$\int_0^T t\|w_t^\varepsilon(t)\|_2^2 dt \leqq C \tag{3.4}$$

where C are various positive constants independent of $\varepsilon \in (0,1]$ and depend on T, Ω, w_0, k and p.

Proof. Multiplying the equation $(NP)_\varepsilon$ by $|w^\varepsilon(x,t)|^{k-1}w^\varepsilon(x,t)$, integrating over Ω and using integration by parts, we have

$$\int w_t^\varepsilon|w^\varepsilon|^{k-1}w^\varepsilon dx + \int (|\nabla w^\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla w^\varepsilon\cdot\nabla(|w^\varepsilon|^{k-1}w^\varepsilon)dx = 0$$

from which we deduce

$$\frac{1}{k+1}\frac{d}{dt}\int |w^\varepsilon|^{k+1}dx + k\int |w^\varepsilon|^{k-1}(|\nabla w^\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}}|\nabla w^\varepsilon|^2 dx = 0 \,. \tag{3.5}$$

Here and from now on, we abbreviate variables x and t of $w(x,t)$ in the integrand.

Since the second term of (3.5) is nonnegative, we have

$$\|w^\varepsilon(t)\|_{k+1} \leqq \|w^\varepsilon(0)\|_{k+1} \leqq \|w_0\|_{k+1}$$

which implies (3.2).

We take $k = 1$ in (3.5) to obtain

$$\frac{1}{2}\frac{d}{dt}\int |w^\varepsilon|^2 dx + \int (|\nabla w^\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}}|\nabla w^\varepsilon|^2 dx = 0$$

from which we have

$$\frac{1}{2}\frac{d}{dt}\int |w^\varepsilon|^2 dx + \int (|\nabla w^\varepsilon|^2 + \varepsilon)^{\frac{p}{2}} dx = \varepsilon\int (|\nabla w^\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} dx \leqq \varepsilon^{\frac{p}{2}}|\Omega| \,. \tag{3.6}$$

Integration (3.6) from 0 to T yields

$$\frac{1}{2}\|w^\varepsilon(T)\|_2^2 + \int_0^T\int (|\nabla w^\varepsilon|^2+\varepsilon)^{\frac{p}{2}}dxdt \leqq |\Omega|T + \frac{1}{2}\|w_0\|_2^2$$

for $\varepsilon \in (0,1]$.

This implies

$$\int_0^T\int |\nabla w^\varepsilon|^p dxdt \leqq \int_0^T\int (|\nabla w^\varepsilon|^2+\varepsilon)^{\frac{p}{2}}dxdt \leqq C\ . \tag{3.7}$$

Multiplying the equation $(NP)_\varepsilon$ by $tw^\varepsilon(x,t)$ and integrating over Ω, we have

$$t\int |w_t^\varepsilon|^2dx + t\int (|\nabla w^\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla w^\varepsilon\cdot\nabla w_t^\varepsilon dx = 0$$

which implies

$$t\int |w_t^\varepsilon|^2dx + \frac{t}{p}\frac{d}{dt}\int (|\nabla w^\varepsilon|^2+\varepsilon)^{\frac{p}{2}}dx = 0\ . \tag{3.8}$$

Integration (3.8) from 0 to T and integration by parts yield

$$\int_0^T t\|w_t^\varepsilon(t)\|_2^2dt + \frac{T}{p}\int (|\nabla w^\varepsilon(x,T)|^2+\varepsilon)^{\frac{p}{2}}dx$$

$$-\frac{1}{p}\int_0^T\int (|\nabla w^\varepsilon(x,t)|^2+\varepsilon)^{\frac{p}{2}}dxdt = 0\ . \tag{3.9}$$

From (3.7) and (3.9), we have

$$\int_0^T t\|w_t^\varepsilon(t)\|_2^2dt \leqq \frac{C}{p}\ .$$

Q.E.D.

Now let w^ε and w^δ be solutions of $(NP)_\varepsilon$ with initial conditions w_0^ε and w_0^δ, respectively.

By the monotonicity of the operator $Au = -\mathrm{div}\left((|\nabla u|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u\right)$, we obtain

$$\|w^\varepsilon(t) - w^\delta(t)\|_{k+1} \leqq \|w_0^\varepsilon - w_0^\delta\|_{k+1}$$

which implies that $\{w^\varepsilon\}$ is a Cauchy sequence in $C([0,T] : L^{k+1}(\Omega))$. Then w^ε tends to w strongly in $C([0,T] : L^{k+1}(\Omega))$.

Moreover, by the well-known argument of the theory of monotone operator (P160, Lions [12]) and (3.4), we obtain

$$\operatorname{div}\left((|\nabla w^\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla w^\varepsilon\right) \to \operatorname{div}(|\nabla w|^{p-2}\nabla w)$$

weakly in $L^2(\Omega \times (\tau,T))$ for any $\tau > 0$.

We conclude that w is a desired solution of $(NP)_2$.

<u>Remark</u>. Since $w(t)$ belongs to $W^{1,p}(\Omega)$ for a.e. $t \in (0,T)$, then $\frac{\partial w}{\partial \nu} = 0$ in the sense that $|\nabla w|^{p-2} \frac{\partial w}{\partial \nu} \in L^{p'}\left(0,T : W^{-\frac{1}{p'},p'}(\Omega)\right)$ where $W^{-\frac{1}{p'},p'}(\Omega)$ is dual of $W^{\frac{1}{p'},p}(\Omega)$ and $\frac{1}{p} + \frac{1}{p'} = 1$. (P165, Lions [12] and Lions-Magenes [13])

We now arrive at the main theorem.

<u>Theorem 3.2</u>.

Let $u(x,t)$ be a solution of (NP) with initial conditions $u_0 \in L^{k+1}(\Omega)$, $k \geq \max(1, \frac{1}{p}(2N - p - Np))$, and $\bar{u}_0 = \frac{1}{|\Omega|} \int u_0(x)dx$.

Then there exists a number $T^* > 0$ such that

$$u(t) \to \bar{u}_0 \quad \text{in} \quad L^{k+1}(\Omega) \quad \text{as} \quad t \to T^* < \infty$$

and

$$u(t) = \bar{u}_0 \quad \text{for} \quad t \geq T^*$$

where T^* is bounded above by $\dfrac{(p + k - 1)^p \|u_0 - \bar{u}_0\|_{k+1}^{2-p}}{Kp^p k(2 - p)}$.

Moreover if $p > \frac{2N}{N + 2}$ or $p = \frac{2N}{N + 2}$ (N is odd), $\|u(t) - \bar{u}_0\|_2$ and $\|\nabla u(t)\|_p$ tend to zero as $t \to T^*$. If $1 < p < \frac{2N}{N + 2}$ or $p = \frac{2N}{N + 2}$ (N is even), there exists a number $T^{**} (\leq T^*)$ such that $\|\nabla u(t)\|_p$ tends to zero as $t \to T^{**}$.

<u>Proof</u>. It is enough to show that the solution w of $(NP)_2$ tends to zero in $L^{k+1}(\Omega)$.

From (3.4) in the proof of Lemma 3.1, we have

$$\frac{1}{k + 1} \frac{d}{dt} \int |w^\varepsilon|^{k+1}dx + k \int |w^\varepsilon|^{k-1}(|\nabla w^\varepsilon|^2 + \varepsilon)^{\frac{p}{2}}dx = \varepsilon^k \int |w^\varepsilon|^{k-1}(|\nabla w^\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} dx .$$

The second term can be rewritten as

$$k\left\{\frac{p^2}{(p+k-1)^2}\,|\nabla(|w^\varepsilon|^{\frac{k-1}{p}}w^\varepsilon)|^2+\varepsilon|w^\varepsilon|^{\frac{2(k-1)}{p}}\right\}^{\frac{p}{2}}$$

which is bounded below by

$$\frac{kp^p}{(p+k-1)^p}\,|\nabla(|w^\varepsilon|^{\frac{k-1}{p}}w^\varepsilon)|^p\ .$$

Then, noting $1 < p < 2$, we have

$$\frac{1}{k+1}\frac{d}{dt}\int |w^\varepsilon|^{k+1}dx+\frac{kp^p}{(p+k-1)^p}\int |\nabla(|w^\varepsilon|^{\frac{k-1}{p}}w^\varepsilon)|^p dx \leqq \varepsilon^{\frac{p}{2}}k\int |w^\varepsilon|^{k-1}dx\ .$$

By Corollary in Section 2, Hölder's inequality and (3.1), we get

$$\frac{1}{k+1}\frac{d}{dt}\int |w^\varepsilon|^{k+1}dx+\frac{kp^pK}{(p+k-1)^p}\left(\int |w^\varepsilon|^{k+1}dx\right)^{\frac{p+k-1}{k+1}} \leqq \varepsilon^{\frac{p}{2}}C \tag{3.10}$$

where C *depends on* k, p, u_0 *and* Ω *but independent of* ε. Let

$$y(t) = \int |w(x,t)|^{k+1}dx\ . \tag{3.11}$$

As ε tends to zero in (3.10), we have the differential inequality

$$y'(t) + C_1 y(t)^{1-\delta} \leqq 0 \quad \text{in} \quad \mathcal{D}'(0,T)$$

where

$$C_1 = \frac{(k+1)p^p kK}{(p+k-1)^p}$$

and

$$0 < \delta = \frac{2-p}{k+1} < 1\ .$$

By the comparison argument and the uniqueness of solutions, we conclude that

$$y(t) \to 0 \quad \text{as} \quad t \to T^*\ ,$$

$$T^* \leq \frac{(p + k - 1)^p \|u_0 - \bar{u}_0\|_{k+1}^{2-p}}{p^p kK(2 - p)}$$

and

$$y(t) \equiv 0 \quad \text{for} \quad t \geq T^* .$$

By the abstract results of [8], we have

$$\|w_t(t)\|_2 \leq \frac{C}{t} \|w_0\|_2$$

for $t > 0$.

From this, we can estimate $\|\nabla u(t)\|_p$ as follows:

$$\begin{aligned}\int |\nabla u|^p dx = \int |\nabla w|^p dx &= -\int \operatorname{div}(|\nabla w|^{p-2}\nabla w) w dx \\ &\leq \|\operatorname{div}(|\nabla w(t)|^{p-2}\nabla w(t))\|_2 \|w(t)\|_2 \\ &= \|w_t(t)\|_2 \|w(t)\|_2 \\ &\leq \frac{C}{t} \|w_0\|_2 \|w(t)\|_2\end{aligned}$$

for $t > 0$.

Since $\operatorname{div}(|\nabla w(t)|^{p-2}\nabla w(t)) \in L^2(\Omega)$ for a.e. $t \in (0,T)$, the first part of this calculation make valid.

From Nirenberg-Galiardo inequality [9], we have

$$\|w(t)\|_r \leq C\|\nabla u(t)\|_p^a \|w(t)\|_1^{1-a}$$

where a satisfies $\frac{1}{r} = (\frac{1}{p} - \frac{1}{N})a + (1 - a)$ and $r \leq \frac{NP}{N - P}$ (the equality holds if N is odd).

If $p > \frac{2N}{N + 2}$ or $p = \frac{2N}{N + 2}$ (N is odd), we have $1 \geq \frac{1}{p}(2p - N - Np)$. Then we can take k as 1 and get

$$\|w(t)\|_2 \leq C_1\|\nabla u(t)\|_p \leq C_2\|w(t)\|_2^{1/p} .$$

In the other case $\left(1 < p < \frac{2N}{N+2}\right.$ or $p = \frac{2N}{N+2}$ (N is even)$\left.\right)$, we get only

$$\|\nabla u(t)\|_p^p \leq \frac{C}{t} \|w(t)\|_2 \leq \frac{C'}{t} \|w(t)\|_{k+1} .$$

4. THE EQUATION $u_t = \Delta_p u + \lambda|u|^{\alpha-1}u$

In this section we will discuss the Neumann problem for the equation

$$u_t = \Delta_p u + \lambda|u|^{\alpha-1}u \tag{4.1}$$

where $\lambda > 0$ and $\alpha \geq 1$.

The equation (4.1) with Neumann zero boundary condition has a solution independent of x which satisfies the ordinary differential equation

$$u_t = \lambda|u|^{\alpha-1}u \quad (u = u(t)) . \tag{4.2}$$

It is well-known that solutions of (4.2) blow up in finite time when $\alpha > 1$ and glow up when $\alpha = 1$. By the comparison theorem, if there exists a positive number δ such that $u_0(x) \geq \delta > 0$ or $0 > -\delta \geq u_0(x)$ for all $x \in \Omega$ ($u_0(x)$ is a initial condition), then the solution of (4.1) with Neumann zero boundary condition and initial condition $u_0(x)$ blow up in finite time when $\alpha > 1$ and glow up when $\alpha = 1$.

We are interested in the case that $u_0(x)$ has positive part and negative part in Ω.

We now give results in the case that $\alpha = 1$. When $\alpha > 1$, it is not yet known whether the problem (4.1) with Neumann zero boundary condition has extinction phenomena or not.

Consider the following problem ($\alpha = 1$)

$$\begin{cases} u_t = \Delta_p u + \lambda u & \text{in } \Omega \times (0,T) \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases} \tag{PP}$$

Let $u(x,t)$ be a solution of (PP) with initial condition $u_0 \in L^{k+1}(\Omega)$, $k \geq \max\left(1, \frac{1}{p}(2N - p - Np)\right)$.

Theorem 4.1.

Let $\bar{u}_0 = \frac{1}{|\Omega|} \int u_0(x)dx$. If $\|u_0 - \bar{u}_0\|_{k+1}$ is sufficiently small, then there exists a number $T^* > 0$ such that

$$u(t) \to \bar{u}(t) \quad \text{in} \quad L^{k+1}(\Omega) \quad \text{as} \quad t \to T^*$$

and

$$u(t) = \bar{u}(t) \quad \text{for} \quad t \geq T^*$$

where $\bar{u}(t) = \bar{u}_0 e^{\lambda t}$.

Moreover T^* is bounded above by

$$\log \left(\frac{p^p kK}{p^p kK - \lambda(p + k - 1)^p \|u_0 - \bar{u}_0\|_{k+1}^{2-p}}\right)^{\frac{1}{(2-p)\lambda}}$$

proof. Integrating the equation (PP) over Ω, we have

$$\frac{d}{dt} \int u(x,t)dx = \lambda \int u(x,t)dx$$

which implies

$$\int u(x,t)dx = |\Omega|\bar{u}_0 e^{\lambda t} \qquad t \in [0,T] .$$

We define

$$w(x,t) = u(x,t) - \bar{u}(t) .$$

Then $w(x,t)$ satisfies

$$\int w(x,t)dx = 0$$

and

$$\begin{aligned} & w_t = \Delta_p w + \lambda w && \text{in } \Omega \times (0,T) \\ & \frac{\partial w}{\partial \nu} = 0 && \text{on } \partial\Omega \times (0,T) \qquad (PP)_1 \\ & w(x,0) = u_0(x) - \bar{u}_0 \equiv w_0(x) && \text{in } \Omega . \end{aligned}$$

Multiplying the equation $(PP)_1$ by $|w(x,t)|^{k-1}w(x,t)$ $(k \geq 1)$, integrating over Ω and through the same procedure as Theorem 3.2, we have

$$\frac{1}{k+1}\frac{d}{dt}\int |w|^{k+1}dx + \frac{p^p kK}{(p+k-1)^p}\left(\int |w|^{k+1}dx\right)^{\frac{p+k-1}{k+1}} \leq \lambda \int |w|^{k+1}dx . \qquad (4.1)$$

Let $y(t) = \|w(t)\|_{k+1}$. From (4.1) we have

$$y'(t) + C_1 y(t)^{p-1} \leq \lambda y(t) \quad \text{in} \quad \mathcal{D}'(0,T) \qquad (4.2)$$

where $C_1 = \frac{p^p kK}{(p + k - 1)^p}$.

By the comparison argument, (4.2) implies

$$y(t) \leq \left[\frac{C_1}{\lambda} - \left(\frac{C_1}{\lambda} - y(0)^{2-p}\right)e^{(2-p)\lambda t}\right]^{\frac{1}{2-p}} . \tag{4.3}$$

If we take u_0 as

$$\|u_0 - \bar{u}_0\|_{k+1} < \left(\frac{C_1}{\lambda}\right)^{\frac{1}{2-p}} = \left(\frac{p^p kK}{\lambda(p + k - 1)^p}\right)^{\frac{1}{2-p}} \tag{4.4}$$

then

$$y(t) \equiv \|w(t)\|_{k+1} \to 0 \quad \text{as} \quad t \to T^*$$

where

$$T^* \leq \log \left(\frac{p^p kK}{p^p kK - \lambda(p + k - 1)^p \|u_0 - \bar{u}_0\|_{k+1}^{2-p}}\right)^{\frac{1}{(2-p)\lambda}}$$

Corollary.

If $\int u_0(x)dx = 0$, that is, $\bar{u}_0 = 0$ and $\|u_0\|_{k+1}$ is sufficiently small, then there exists a number $T^* > 0$ such that

$$u(t) \to 0 \quad \text{in} \quad L^{k+1}(\Omega) \quad \text{as} \quad t \to T^*$$

and

$$u(t) \equiv 0 \quad \text{for} \quad t \geq T^* .$$

Remark.

Consider the stationary problem for (PP):

$$\begin{aligned} -\Delta_p u &= \lambda u \quad && \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on} \quad \partial\Omega . \end{aligned} \tag{SP}$$

If (SP) has a solution, we have

$$\frac{p^p k}{(p + k - 1)^p} \int |\nabla(|u|^{\frac{k-1}{p}} u)|^p dx = \lambda \int |u|^{k+1} dx .$$

Since $\int u(x)dx = 0$, we can apply Corollary in Section 2 to obtain

$$\frac{p^p k K}{(p + k - 1)^p} \left(\int |u|^{k+1} dx\right)^{\frac{p+k-1}{k+1}} \leq \lambda \int |u|^{k+1} dx$$

for $k \geq \max\left(1, \frac{1}{p}(2N - p - Np)\right)$ which implies

$$\left(\frac{p^p k K}{(p + k - 1)^p \lambda}\right)^{\frac{1}{2-p}} \leq \|u\|_{k+1} . \tag{4.5}$$

Comparing (4.4) with (4.5), we say if $\|u_0\|_{k+1}$ is larger than $\left(\frac{p^p k K}{(p + k - 1)^p \lambda}\right)^{\frac{1}{2-p}}$, then there exists a solution of (PP) which is independent of t for a special initial condition $u_0(x)$, and this solution does not extinct in finite time whenever $\int u_0(x)dx = 0$.

From this point, we conclude that the condition (4.4) for the homogenization (and extinction) of u is critical.

REFERENCES

[1] N. D. Alikakos, Degenerate homogeneous equations.

[2] N. D. Alikakos and R. Rostamian, Gradient estimates for degenerate diffusion equaiton. I, Math. Ann., 259 (1982) 53-70.

[3] ______________, Gradient estimates for degenerate diffusion equation. II, Proc. Royal Soc. Edinburgh 91A (1982) 335-346.

[4] ______________, Lower bound estimates and separable solutions for homogeneous equations of evolution in Banach space, J. Diff. Eq., 42 (1982) 323-344.

[5] N. D. Alikakos and L. C. Evans, Continuity of the gradient for weak solutions of a degenerate parabolic equation, J. Math. pures et appl., 62 (1983) 235-268.

[6] J. G. Berryman and C. B. Holland, Stability of the separable solution for fast diffusion, Arch Rat. Mech. Anal., 74 (1980) 379-388.

[7] E. DiBenedetto, Continuity of weak solutions to certain singular parabolic equations, MRC Tech. Summary Report #2124 (1980).

[8] M. G. Crandall and P. Bénilan, Regularizing effect of homogeneous evolution equations, MRC Tech. Summary Report #2076 (1980).

[9] A. Friedman, "Partial differential equations" Holt Rienhart and Winston, Inc., New York (1969).

[10] D. Gilbarg and N. S. Trudinger, "Elliptic partial differential equations of second order", Springer-Verlag, Berlin, Heidelberg (1977).

[11] O. A. Ladyzenskaya, V. A. Sollonnikov and N. N. Ural'ceva, "Linear and quasilinear equations of parabolic type" Moscow Nauka (1967); English Transl. Math. Monographs vol. 23, Providence, R.I. AMS (1968).

[12] J. L. Lions, "Quelques méthodes de résolution des problèmes aux limites nonlinéaires, Dunud Ganthier-Villars, Paris (1969).

[13] J. L. Lions and E. Magenes, Problèmes aux limites non homogènes (IV) 15 (1961) 311-326.

[14] M. Otani, Nonmonotone perturbation for nonlinear parabolic equaitons associated with subdifferential operators, periodic problems, J. Diff. Eq., 54 (1984) 248-273.

[15] E. S. Sabinina, A class of nonlinear degenerating parabolic equations, Sov. Math. Doklady 143 (1962) 495-498.

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It is shown that solutions of the Neumann problem

$$u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) \quad \text{in} \quad \Omega \times (0,T)$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0,T)$$

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega$$

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ABSTRACT (cont.)

tend to some constant solutions in finite time, where $1 < p < 2$ and Ω is a bounded domain in $\mathbb{R}^N$.

In order to prove this, we establish Sobolev-Poincaré's inequality for functions in $W^{1,p}(\Omega)$ under some assumptions.

We treat the extinction phenomena for the equation $u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda u \quad (\lambda > 0)$ with Neumann boundary conditions.

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